§5.6 Matrix Exponentials / Linear Systems

the solution vectors of an $n \times n$ non-HG linear system $\vec{x}' = A\vec{x}$ can be used to construct a square matrix $X = \phi(t)$ that satisfies $X' = AX$

suppose $\vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t)$ are n linearly independent sol'ns of $\vec{x}' = A\vec{x}$

then can use to construct $n \times n$ matrix ϕ where each sol'n is a column vector

$$\phi(t) = \begin{bmatrix} | & | & & | \\ \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \\ | & | & & | \end{bmatrix}$$

called fundamental matrix for $\vec{x}' = A\vec{x}$

gen. soln in "LC" format:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \dots + c_n\vec{x}_n(t) \quad \leftarrow \text{expanded}$$

condense: $\vec{x} = \phi(t)\vec{c}$ where $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ ← arb. constant vector

key takeaway: ϕ is invertible ← has an inverse matrix ϕ^{-1}

given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $\det A = ad - bc \neq 0$

then A^{-1} exists

and $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

ex. find a fundamental matrix for system

$$\begin{aligned} x' &= 4x + 2y \\ y' &= 3x - y \end{aligned} \qquad \text{let } \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\vec{v}' = A\vec{v}$$

$$\vec{v}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

char eqn:

det $(A-\lambda I)=0$

$$\det\begin{bmatrix}4-\lambda & 2\\ 3 & -1-\lambda\end{bmatrix}=0$$

$$(4-\lambda)(-1-\lambda)-6=0$$

$$\lambda^2-3\lambda-10=0$$

$$(\lambda-5)(\lambda+2)=0$$

$$\lambda_1=5 \quad \lambda_2=-2$$

when $\lambda=5$

$$\begin{bmatrix}-1 & 2\\ 3 & -6\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$-x+2y=0$$

$$x=2y \qquad EV_{\lambda=5}\begin{bmatrix}2\\ 1\end{bmatrix}$$

$$y=y$$

$$\vec{x}_1(t)=\begin{bmatrix}2e^{5t}\\ e^{5t}\end{bmatrix}$$

$\lambda=-2$

$$\begin{bmatrix}6 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$3x+y=0$$

$$y=-3x$$

$$x=x$$

$$EV_{\lambda=-2}\begin{bmatrix}1\\ -3\end{bmatrix}$$

$$\vec{x}_2(t)=\begin{bmatrix}e^{-2t}\\ -3e^{-2t}\end{bmatrix}$$

$$\phi(t)=\begin{bmatrix}2e^{5t} & e^{-2t}\\ e^{5t} & -3e^{-2t}\end{bmatrix}$$

Initial Conditions:

thm: let $\phi(t)$ = fundamental matrix for HG linear system $\vec{x}'=A\vec{x}$
then unique soln of IVP $\vec{x}'=A\vec{x}$ given $\vec{x}(0)=\vec{v}$
is $\vec{x}(t)=\phi(t)\phi^{-1}(0)\vec{x}(0)$

back to previous example given $x(0)=1 \quad y(0)=-1$

$$\phi(0)=\begin{bmatrix}2 & 1\\ 1 & -3\end{bmatrix} \qquad \det\phi(0)=-6-1=-7$$

$$\phi^{-1}(0)=-\frac{1}{7}\begin{bmatrix}-3 & -1\\ -1 & 2\end{bmatrix}$$

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad A^{-1}\ \frac{1}{\det A}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

$\vec{v}$ from prev pg.

$7 \begin{bmatrix} -1 & 2 \end{bmatrix}$ ... det A

$\xleftarrow{} \vec{v}$ from prev pg.

$$\vec{x}(t) = \phi(t)\,\phi^{-1}(0)\,\vec{x}(0)$$

$$= -\frac{1}{7}\begin{bmatrix} 2e^{5t} & e^{-2t} \\ e^{5t} & -3e^{-2t} \end{bmatrix}\underbrace{\begin{bmatrix} -3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$= \phi \begin{bmatrix} -3+1 \\ -1 & -2 \end{bmatrix}$$

$$= -\frac{1}{7}\,\phi \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$= \underbrace{\frac{1}{7}\begin{bmatrix} 2e^{5t} & e^{-2t} \\ e^{5t} & -3e^{-2t} \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$$\vec{x}(t) = \frac{1}{7}\begin{bmatrix} 4e^{5t} + 3e^{-2t} \\ 2e^{5t} - 9e^{-2t} \end{bmatrix} \xrightarrow{\text{as eqns}} \boxed{\begin{aligned} x(t) &= \tfrac{4}{7}e^{5t} + \tfrac{3}{7}e^{-2t} \\ y(t) &= \tfrac{2}{7}e^{5t} - \tfrac{9}{7}e^{-2t} \end{aligned}}$$

$n \times n$ matrix multiplication

given $A = [a_{ij}]$ $\quad B = [b_{ij}]$ $\qquad i$ = ith row, j = jth column

$$A = \begin{bmatrix} a_{11} & & & \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & & & \\ \vdots & \cdots & \cdots & a_{mp} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1j} & & \\ & & b_{2j} & & \\ & & \vdots & & \\ & & b_{pj} & \cdots & b_{pn} \end{bmatrix}$$

$A = m \times p$ matrix $\qquad B = p \times n$ matrix

$$C = AB \quad [a_i \cdot b_j]$$

entry $c_{ij} = \sum_{k=1}^{p} a_{ik} \cdot b_{kj}$ $\qquad C = m \times n$ matrix

Exponential Matrices

A is a 2×2 diagonal matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$$A^2 = AA = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

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$$= \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix}$$

logical that $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$

recall power series $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} \cdots + \frac{z^n}{n!} + \cdots$

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

then exponential matrix e^A is an $n \times n$ defined as:

$$e^A = I + A + \frac{A^2}{2!} + \cdots \frac{A^n}{n!} + \cdots$$

matrix

$e^0 = I$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} \frac{a^2}{2!} & 0 \\ 0 & \frac{b^2}{2!} \end{bmatrix} + \cdots$$

add each entry to form 1 matrix

$$= \begin{bmatrix} \overbrace{1 + a + \frac{a^2}{2!} + \cdots}^{e^a} & 0 \\ 0 & \underbrace{1 + b + \frac{b^2}{2!} + \cdots}_{e^b} \end{bmatrix}$$

$$e^A = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

expand concept to $n \times n$ diagonal matrices:

$$D = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}$$

then $e^{D} \begin{bmatrix} e^{a_1} & & 0 \\ & e^{a_2} & \\ 0 & & \ddots \\ & & & e^{a_n} \end{bmatrix}$

fact: matrices are not automatically commutable $AB \neq BA$ (not nec)

if they are commutable $(AB = BA)$ then $e^{A+B} = e^{A}e^{B}$ holds

let t = scalar variable

then $e^{At} = \underset{n=0}{I} + \underset{n=1}{At} + \underset{n=2}{A^2 \frac{t^2}{2!}} + \ldots + A^n \frac{t^n}{n!} + \ldots$

when A is not a diagonal matrix:

ex. given $A = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

then $A^2 = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

conclude A^n is zero matrix for $n \geq 3$

$\therefore e^{At}$ is finite or nilpotent

$e^{At} = I + At + \frac{1}{2}A^2t^2$

$= \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} + \overset{A}{\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}} t + \frac{1}{2} \overset{A^2}{\begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} t^2$

$e^{At} = \begin{bmatrix} 1 & 3t & 4t + 9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{bmatrix}$ given $A = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

same

ex. given $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix} = D + B$

$\begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & 4 \end{bmatrix}$

ex given $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix} = D + B$ ← same

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

then $e^{At} = e^{(D+B)t} = e^{Dt} e^{Bt}$

$$\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 3t & 4t+9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{bmatrix}$$

multiply

$$e^{At} = \begin{bmatrix} e^{2t} & 3te^{2t} & (4t+9t^2)e^{2t} \\ 0 & e^{2t} & 6t\, e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$\frac{d}{dt} e^{At} = \frac{d}{dt}\left(I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \dots\right)$$

$$= A + A^2\frac{2t}{2!} + A^3\frac{3t^2}{3!} + \dots \qquad \frac{2}{2!} = \frac{\cancel{2}}{\cancel{2}\cdot 1} \qquad \frac{3}{3!} = \frac{\cancel{3}}{\cancel{3}\cdot 2\cdot 1}$$

$$= A + A^2 t + A^3\frac{t^2}{2!} + \dots \quad \text{factor out } A$$

$$= A\left(I + At + A^2\frac{t^2}{2!} + \dots\right)$$

$$\frac{d}{dt} e^{At} = A\, e^{At}$$

can say then that the matrix-valued function $X(t) = e^{At}$ is a sol'n to $X' = AX$

given that e^{At} is invertible, then it's a fundamental matrix for linear system $\vec{x}' = A\vec{x}$ s.t. $X(0) = I$

thm: given $A =$ $n \times n$ matrix then $\vec{x}' = A\vec{x}$ has unique sol'n with $\vec{x}(0) = \vec{x}_0$ ~~is~~ in format: $\vec{x}(t) = e^{At}\vec{x}_0$

saw earlier $\vec{x}(t) = \underbrace{\phi(t)\,\phi^{-1}(0)}_{e^{At}}\,\underbrace{\vec{x}(0)}_{\vec{x}_0}$

can find e^{At} by solving $\vec{x}' = A\vec{x}$

revisit $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ has $\phi(t) = \begin{bmatrix} 2e^{5t} & e^{-2t} \\ e^{5t} & -3e^{2t} \end{bmatrix}$

also $\phi^{-1}(0) = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$ ← extraneous negatives canceled out

then $e^{At} = \frac{1}{7}\begin{bmatrix} 2e^{5t} & e^{-2t} \\ e^{5t} & -3e^{2t} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$